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A model for subharmonic resonance within wavepackets in unstable boundary layers

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In recent experiments on the growth of localized disturbances in a Blasius boundary layer, Medeiros & Gaster (1999*a, b*) observed that the development of nonlinear effects depends markedly on the initial phase of their imposed disturbance. Here, a simple explanation of this phenomenon is proposed. Because the disturbance is localized in space and time, it has a spread of wavenumbers and frequencies: among these are components which can initiate a pair of resonant subharmonic waves with well-determined phase, which are then amplified by the familiar three-wave resonance mechanism. The amplitude attained after some time is strongly phase-dependent, consistent with the experimental observations.

1. Introduction

Two recent papers of Medeiros & Gaster (1999*a, b*) have extended and refined the well-known experiments of Gaster & Grant (1975) on the growth of small localized disturbances in an unstable Blasius boundary layer. During the initial period of linear evolution, the form of disturbances was found to be insensitive to the initial phase of their excitation pulse, apart from the expected phase change. However (as shown in figure 1 below), nonlinear distortion of the pulse at larger amplitudes was strongly affected by the initial phase of excitation: for instance, 'positive' pulses (with ejection of fluid) exhibited nonlinear effects at lower amplitudes than did 'negative' pulses (with withdrawal of fluid). Their data suggest that three-wave subharmonic resonance, of the sort studied theoretically by Craik (1971), is significantly involved in the nonlinear growth. However, due to the localized nature of the evolving wavepacket, with a consequent spread of wavenumbers, the situation is less clear-cut than in the vibrating-ribbon experiments of Kachanov & Levchenko (1984) and of Saric & Thomas (1984). (In particular, the frequencies of nonlinearly excited waves are close to, rather than precisely equal to, the half-frequency of the largest two-dimensional mode.)

Medeiros & Gaster (1999*b*) convincingly argue that random external disturbances do not play any part in their results, since the phase-dependence is a consistently repeatable phenomenon. Accordingly, they performed a careful study to determine the importance, in their initial pulse, of the 'most dangerous' components at subharmonic frequencies. On repeating their experiments with these components entirely *removed* from the initial disturbance, they found that their results were hardly altered. This surprising fact raises interesting questions. As their results are deterministic, but independent of the imposed subharmonic frequency components, where does the 'seed' for the subharmonic come from, if three-wave resonance is to play a role? And why does the phase of the initial disturbance play such an important part? In the words of Medeiros & Gaster (1999*b*, p. 316), '... the deterministic oblique modes must have come from some as yet unidentified mechanism of wave production'. The present note outlines such a mechanism. (The ideas described here originate in preliminary unpublished work, done long ago, in an attempt to understand the 'warping' of wavepackets in the experiments of Gaster & Grant (1975): the new experiments have brought this into sharper focus.)

2. Nonlinear wave forcing

Suppose that there are two constant-amplitude plane waves, with fluctuating amplitudes represented according to linear theory by

$$A_1 = a_1 \exp[\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{x} - \omega_1 t)], \quad A_2 = a_2 \exp[\mathrm{i}(\boldsymbol{k}_2 \cdot \boldsymbol{x} - \omega_2 t)],$$

in standard notation. As is well known, their quadratic interaction gives two second-order components proportional to A_1A_2 and $A_1A_2^*$, where $(^*)$ denotes complex conjugate, with respective wavenumbers $\boldsymbol{k}_1 + \boldsymbol{k}_2$, $\boldsymbol{k}_1 - \boldsymbol{k}_2$ and frequencies $\omega_1 + \omega_2$, $\omega_1 - \omega_2$. Cases of three-wave resonance occur when one or other of these frequencies is close to that of a linear wavemode with the appropriate wavenumber $\boldsymbol{k}_1 \pm \boldsymbol{k}_2$: it is then possible for the new mode to grow to amplitudes comparable with that of the two original waves (see e.g. Craik 1985). But, in all non-resonant cases, the forced quadratic disturbance remains relatively small. When the wavemodes A_1, A_2 themselves undergo linear amplification or damping with time t, their frequencies are complex quantities with imaginary parts representing the exponential growth or damping rates. Provided the growth or decay rates are small, the condition for near-resonance is much as above, but involving only the real parts of the frequencies.

In boundary-layer instability, the situation is not quite so simple. Plane and oblique Tollmien–Schlichting waves may have spatial as well as temporal growth or decay rates; and these, together with the wavenumbers, slowly change as the waves propagate downstream, owing to the increasing boundary-layer thickness. However, for present purposes, it is sufficient to confine attention to purely temporal growth, and to suppose that the flow is a uniform parallel shear flow without streamwise dependence. (Of course, this would not do for a precise quantitative model.)

With coordinates x downstream, y spanwise on the plate, and z normal to the plate, let the two linear modes A_1, A_2 have wavenumbers $\boldsymbol{k}_1 = (k_1, l_1)$, $\boldsymbol{k}_2 = (k_2, l_2)$ in the (x, y)-plane. The dependence of their velocity components on z is governed by a generalized Orr–Sommerfeld equation and an ancillary equation for velocity fluctuations parallel to constant-phase lines: see e.g. Craik (1985, pp. 166–167). Their quadratic interaction of form $A_1A_2^*$ enters a (weakly) nonlinear equation for each mode A_3 with wavenumber $\boldsymbol{k}_1 - \boldsymbol{k}_2$. This has the general form

$$(\mathrm{d}/\mathrm{d}t + \mathrm{i}\,\omega_3)A_3 = \sigma_{12}A_1A_2^*, \tag{2.1}$$

where σ_{12} is a complex constant which is expressible as an integral involving the linear eigenfunctions and adjoint eigenfunctions of the three wavemodes: see e.g. Craik (1985, p. 68). In fact, there is both a discrete and a continuous spectrum of modes A_3 for each wavenumber, each with its own structure in z and complex linear eigenvalue ω_3; but most of these are rapidly damped, and here it is sufficient to focus on the least-damped mode corresponding to an oblique Tollmien–Schlichting wave.

On writing $A_3 = a_3 \exp[\mathrm{i}(\boldsymbol{k}_3 \cdot \boldsymbol{x} - \omega_3 t)]$ and reinstating the values for the

ω_j $(j = 1, 2, 3)$ known from linear theory, one may integrate to obtain

$$a_3 = \frac{\sigma_{12} a_1 a_2^* \exp\left[-\mathrm{i}(\omega_{1r} - \omega_{2r} - \omega_{3r})t\right] \exp\left[(\omega_{1i} + \omega_{2i} - \omega_{3i})t\right]}{(\omega_{1i} + \omega_{2i} - \omega_{3i}) - \mathrm{i}(\omega_{1r} - \omega_{2r} - \omega_{3r})} + \text{const.} \tag{2.2a}$$

Here, the subscripts r and i denote real and imaginary parts. This procedure is valid as long as the denominator is non-zero and $|A_3|$ remains sufficiently small that it does not influence the waves A_1 and A_2. Even close to three-wave resonance, result (2.2*a*) may provide a satisfactory approximation during the initial stage of generation of an A_3 mode in the presence of pre-existing modes A_1 and A_2. However, if the denominator of (2.2*a*) is exactly zero, the appropriate solution is

$$a_3 = \sigma_{12} a_1 a_2^* t + \text{const.} \tag{2.2b}$$

In contrast, the subharmonic resonance mechanism of Craik (1971) envisages a symmetric triad of waves with wavenumbers $\boldsymbol{k}_0 = (k, 0)$, $\boldsymbol{k}_+ = (k/2, l)$, $\boldsymbol{k}_- = (k/2, -l)$, where $\boldsymbol{k}_0$ is the initially dominant mode selected by linear amplification, and $\boldsymbol{k}_+$, $\boldsymbol{k}_-$ are subharmonic modes (usually lightly damped) with equal frequencies $\omega_\pm$ having real part very close to half that of the dominant mode. The last requirement means that $\pm l$ must lie fairly close to a particular value which ensures resonance. For Blasius flow, the corresponding wavenumbers $\boldsymbol{k}_+$ and $\boldsymbol{k}_-$ are typically inclined to the mean-flow direction at equal and opposite angles somewhere between 40° and 60°. As the frequency varies rather slowly with l, but relatively rapidly with k, sharp selection of l by the resonance criterion is not to be expected. Slightly detuned modes with values of $\pm l$ different from the resonant value are also amplified: see Kachanov & Levchenko (1984) and Kachanov (1994). For precise resonance, the quadratic equations for the amplitudes a_0, a_+ and a_- are given by Craik (1985, p. 168) and Medeiros & Gaster (1999*b*, p. 311), with straightforward extension to near-resonant cases: see e.g. Zelman & Maslennikova (1993). When the subharmonic modes have sufficiently small amplitude compared with the fundamental mode a_0, as assumed in this paper, the alternative linear Floquet analysis of Herbert (1984) is equally applicable.

In the notation of (2.1), the dominant mode $\boldsymbol{k}_0$ is identified with $A_1 \equiv A_0$ say, and the subharmonics $\boldsymbol{k}_+, \boldsymbol{k}_-$ with $A_2 \equiv A_+$ and $A_3 \equiv A_-$. If, initially, both A_+ and A_- are infinitesimally small, a solution of type (2.2*a*) is inappropriate, since this was based on the supposition of a small third mode A_3 driven by two dominant modes A_1 and A_2. Instead, with a single dominant mode A_0, both A_+ and A_- together grow from infinitesimal levels. If, for simplicity, we suppose that the respective *linear* growth rate ω_l of all three modes is zero, then the complex amplitudes a_+ and a_- of A_+ and A_- are governed by the equations

$$\mathrm{d}a_+/\mathrm{d}t = \lambda a_0 a_-^*, \quad \mathrm{d}a_-/\mathrm{d}t = \lambda a_0 a_+^*, \tag{2.3}$$

where the complex amplitude a_0 of mode A_0 may be taken as constant, and λ is a constant complex coefficient (cf. Medeiros & Gaster 1999*b*, p. 311, with their $\sigma_j = 0$). For situations symmetric about the x-axis, as envisaged here, a_- and a_+ must be equal. The initial evolution of small a_+ and a_- is then a sum of growing and decaying exponentials, with the form

$$a_+ = a_- = \mathrm{e}^{\mathrm{i}\kappa}(p_1 \mathrm{e}^{|\lambda a_0| t} + \mathrm{i} p_2 \mathrm{e}^{-|\lambda a_0| t}), \tag{2.4}$$

where p_1 and p_2 are arbitrary *real* constants and $\kappa \equiv \mathrm{ph}(\lambda a_0)/2$. The corresponding initial complex value of a_+ and a_- is $\exp(\mathrm{i}\kappa)(p_1 + \mathrm{i}p_2)$: if this initial value is known, then p_1 and p_2 may easily be determined. The amplification factor of a_+ and a_- after time t is

$$\frac{|p_1 e^{|\lambda a_0|t} + i p_2 e^{-|\lambda a_0|t}|}{|p_1 + i p_2|} \approx e^{|\lambda a_0|t} \frac{|p_1|}{|p_1 + i p_2|} \quad \text{for} \quad |\lambda a_0|t \gg 1, \quad p_1 \neq 0.$$

This factor clearly depends on the relative magnitudes of p_1 and p_2: from (2.4), this is seen to be greatest when the initial phase of a_+^2 and a_-^2 equals that of λa_0 (i.e. when $p_2 = 0$), and it approaches zero when the initial phase of a_+^2 and a_-^2 differs by 180° from that of λa_0 (i.e. when $p_1 \to 0$). Such phase-dependent amplification of subharmonic modes is demonstrated in the recent experiments of Bake, Fernholz & Kachanov (2000): see also Kachanov (1994). This has important implications, discussed below, for the development of nonlinear localized disturbances in the experiments of Medeiros & Gaster (1999*a*, *b*).

3. A simple model

First, we consider an idealized and over-simplified model of a wavepacket comprising just five waves, with wavenumbers $\boldsymbol{k}_0 = (k, 0)$, $\boldsymbol{k}_1 = (\boldsymbol{k} + \Delta, l/2)$, $\boldsymbol{k}_2 = (k + \Delta, -l/2)$, $\boldsymbol{k}_3 = (k - \Delta, -l/2)$, $\boldsymbol{k}_4 = (k - \Delta, -l/2)$. Suppose that $\boldsymbol{k}_0$ is the linearly most-unstable mode of the packet, and that the others are representatives of all other available linearly unstable modes with wavenumbers centred around $\boldsymbol{k}_0$. The difference interaction of $\boldsymbol{k}_1$ and $\boldsymbol{k}_4$ produces quadratic terms with wavenumber $(2\Delta, l)$; and the similar interaction of $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ produces terms with wavenumber $(2\Delta, -l)$. Let the real part of the frequency of modes $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ be ω_{1r} say; and let that of modes $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ be $\omega_{1r} - \delta$. Then, the above difference interactions drive wavenumbers $(2\Delta, \pm l)$ with frequency δ.

If it should happen that $2\Delta = k/2$ and $\delta = \omega_{0r}/2$, these components will correspond precisely with the subharmonic modes $\boldsymbol{k}_+$ and $\boldsymbol{k}_-$ which interact resonantly with the linearly most-unstable mode $\boldsymbol{k}_0$ with (real) frequency ω_{0r}. In that event, the *initial selection* of the resonant subharmonics is made by the above quadratic interactions. But, since the amplitude of the most-unstable mode $\boldsymbol{k}_0$ is usually considerably larger than that of the other waves $\boldsymbol{k}_j (j = 1, 2, 3, 4)$, the subsequent resonant growth of the subharmonics should then follow roughly in accord with (2.3), but with the arbitrary real constants p_1 and p_2 which appeared in the previous section now pre-determined by the initial quadratic forcing.

However, we must proceed rather less intuitively. If we impose the same restrictive assumption made above, that the linear growth or decay of participating modes may be neglected, and retain all relevant quadratic terms but neglect all terms of higher order, equations (2.3) are replaced by

$$da_+/dt = \lambda a_0 a_-^* + \mu a_1 a_3^*, \quad da_-/dt = \lambda a_0 a_+^* + \mu a_1 a_3^*, \tag{3.1}$$

where λ and μ correspond to the respective coefficients σ_{ij} of the participating modes, as introduced at (2.1) above. Here, we have also invoked symmetry with respect to y, setting the constant complex amplitudes of both modes $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ equal to a_1, and those of $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ equal to a_3. (More formally, one may take all five participating mode amplitudes to be $O(\varepsilon)$, where ε is a small parameter, which depend on the slow time variable $\tau = \varepsilon t$: one then neglects all terms of higher order than $O(\varepsilon^2)$. This procedure remains valid even when a_- and a_+ are initially zero.)

Taking $a_- = a_+$ as before, and introducing the transformation $a_- = a_+ \equiv b - (\mu a_1 a_3^* / \lambda a_0)^*$ yields the equation

$$db/dt = \lambda a_0 b^*, \tag{3.2}$$

which has the same solution for $b(t)$ as that given in (2.4) for a_+ and a_-. Accordingly, the extra constant forcing terms shift the effective origin of a_+ and a_-. The solution having $a_- = a_+ = 0$ at time $t = 0$ is that with $b(0) = (\mu a_1 a_3^*/\lambda a_0)^*$. But the general solution for $b(t)$ is just that of (2.4), and the initial condition $a_- = a_+ = 0$ requires that

$$p_1 + \mathrm{i}p_2 = \mathrm{e}^{\mathrm{i}\kappa}(\mu a_1)^* a_3 |\lambda a_0|^{-1}. \tag{3.3}$$

Accordingly, a_- and a_+ are known for all times $t > 0$.

Now, since $a_- = a_+ = 0$ initially, this solution has $\mathrm{d}a_\pm/\mathrm{d}t$ initially equal to $\mu a_1 a_3^*$: that is to say, the solution with initial growth from zero is at first driven by the interaction of modes a_1 and a_3 alone. This is just as envisaged in (2.2*b*) (but now with $\sigma_{12} = \mu$ and suffices 3, 2 replaced by $\pm$, 3), and in the remarks at the start of this section. At later times such that $|\lambda a_0|t \gg 1$ (but not so large that a_0 is itself nonlinearly affected), the resonant subharmonic interaction with a_0 becomes dominant, and the 'amplification factor' given below (3.4) is

$$\mathrm{e}^{|\lambda a_0|t}\frac{|p_1|}{|p_1 + ip_2|} = \mathrm{e}^{|\lambda a_0|t}|\cos[\tfrac{1}{2}\mathrm{ph}(\lambda a_0) - \mathrm{ph}(\mu a_1 a_3^*)]|. \tag{3.4}$$

However, this amplification factor is for $|b(t)|$, not $|a_\pm(t)|$. Although still a good measure of the dependence of the disturbance size on the phases of the participating waves after some time has elapsed, a more appropriate quantity is the asymptotic value of $|a_\pm(t)|$, which is

$$|a_\pm| \approx \mathrm{e}^{|\lambda a_0|t}\frac{|\mu a_1 a_3|}{|\lambda a_0|}|\cos[\tfrac{1}{2}\mathrm{ph}(\lambda a_0) - \mathrm{ph}(\mu a_1 a_3^*)]|. \tag{3.5}$$

Thus, through the cosine term, the growth of the subharmonic is influenced by the phases of the members of the initial wavepacket, in qualitative agreement with the experimental findings of Medeiros & Gaster (1999*a, b*); but closer examination is necessary before concluding that the correct explanation has been found.

Of course, this model is a gross over-simplification of the experimental situation, and several questions must now be addressed. (i) Do the experiments admit linearly unstable modes with the wavenumber and frequency characteristics postulated above? (ii) If so, can an extended version of the above model be constructed to take account of a multiplicity of modes? (iii) To what extent is one justified in ignoring the linear growth or decay rates of the participating modes during the period of nonlinear evolution?

4. Multi-mode wavepackets

In practice, a localized wavepacket consists of very many Fourier modes, many pairs of which have the potential to drive up a subharmonic mode, in much the same way as the wave pairs $\boldsymbol{k}_1$, $\boldsymbol{k}_4$ and $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ of the previous section. A necessary requirement is that the linearly unstable bands of frequency and wavenumber are sufficiently wide to admit a difference interaction which supports the subharmonic modes. Among Medeiros & Gaster's (1999*b*) experimental data are graphical descriptions of their developing wavepackets in the frequency versus spanwise-wavenumber plane. They suggest that a subharmonic resonance mechanism (such as is modelled above by equation (2.3)) may operate at downstream distances greater than their $x = 0.5$ m, but that some other unknown deterministic mechanism must first initiate the small subharmonics somewhere between $x = 0.3$ m and 0.5 m.

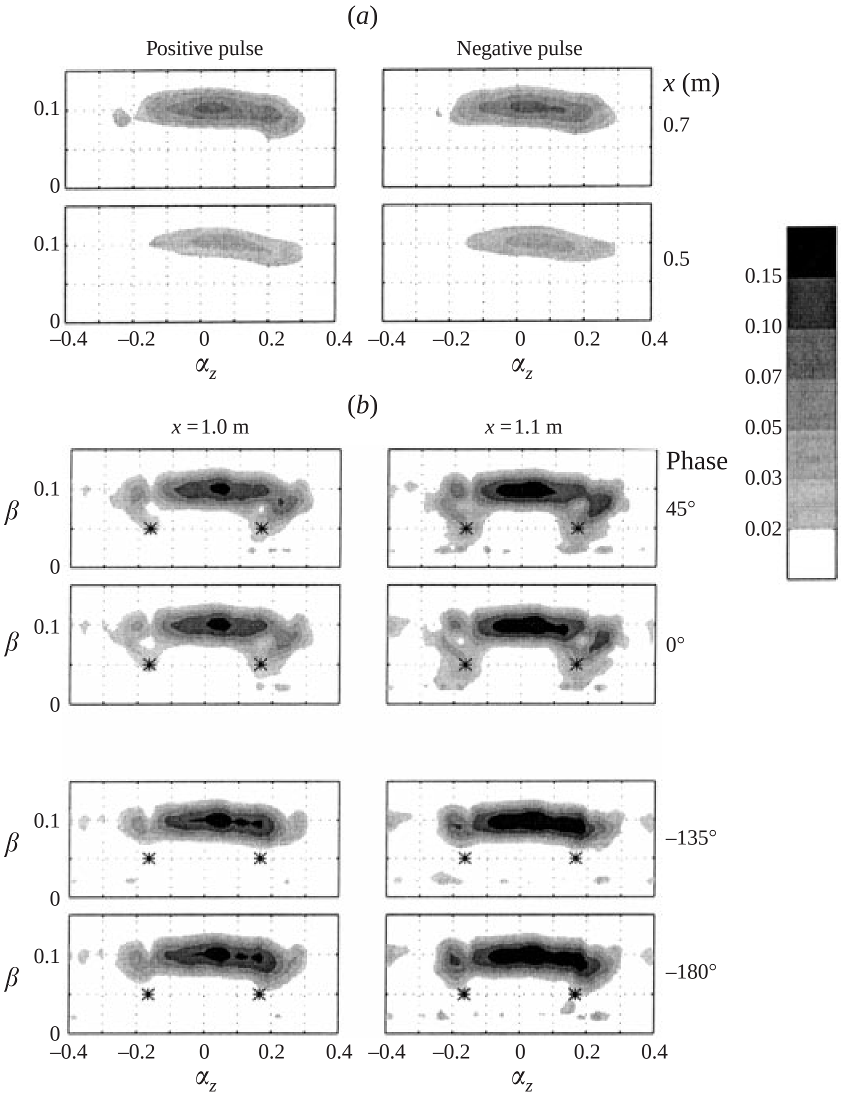

FIGURE 1. Some experimental wavepackets of Medeiros & Gaster (1999*b*), in terms of frequency β and spanwise wavenumber α_z, with relative amplitudes indicated by the grey-scale. (*a*) Near-identical wavepackets with positive initial pulse (phase $= 0°$) and negative initial pulse (phase $- 180°$) at downstream distances $x = 0.5$ m and 0.7 m. (*b*) Wavepackets farther downstream, at $x = 1.0$ m and 1.1 m, showing the effect of nonlinearity for different initial phases 45°, 0°, −135°, −180°. Asterisks indicate the locations of resonant subharmonics. (Reproduced, with permission, from Medeiros & Gaster's figures 3 and 4.)

Taken from Medeiros & Gaster (1999*b*, figure 3), wavepacket graphs of spanwise wavenumber (their α_z) versus frequency (their β) at $x = 0.5$ m and 0.7 m are shown in figure 1(*a*) for both positive and negative pulses. Clearly, there is little difference between positive and negative cases. The grey-scale represents the relative magnitudes of the wavemodes. But, further downstream, the phase of the initial disturbance becomes increasingly important. Figure 1(*b*) shows some of Medeiros & Gaster's data (from their figure 4) at $x = 1.0$ m and 1.1 m, for initial phases of 0°, 45°, −180° and

-135°. A positive pulse has phase of 0° and a negative one -180°: for a description of how the initial phase was controlled, see Medeiros & Gaster (1999*a*, p. 270). Asterisks denote the theoretical location of the subharmonic modes which resonate with the strongest two-dimensional mode. The growth of near-subharmonic modes is evidently far stronger for phases 0° and 45° than for -180° and -135°.

The frequency parameter β is our ω_r, and the spanwise-wavenumber parameter α_z is our l. The packets at small x-values are roughly elliptical or 'cigar-shaped', with minor and major axes located at $0.075 < \beta < 0.125$ and $-0.15 < \alpha_z < 0.3$ approximately, for $x = 0.5\,\mathrm{m}$. (The asymmetry with respect to α_z is surprising and unexplained; but further downstream this becomes less marked. Below, we examine a symmetric model with $-0.24 \leqslant \alpha_z = l \leqslant 0.24$.) The most amplified mode is located near $\beta = 0.1$ at a small value of α_z which we shall take to be zero in accord with theory. The subharmonics which resonate with the most-amplified mode have $\beta = 0.05$ and $\alpha_z = 0.17$ approximately. This value of $\beta = 0.05$ is *almost exactly equal* to the maximum recorded width of the packet in the β-direction at $x = 0.5\,\mathrm{m}$: this does not seem to be a mere coincidence, for this is the minimum width necessary to admit pairs of linear modes capable of generating the subharmonic by difference interaction, as outlined in the preceding section. In fact, since the lowest recorded amplitude of Medeiros & Gaster was 0.02 units, the full bandwidth of the developing packet predicted by linear stability theory is somewhat larger than that recorded; therefore, the actual width must be slightly larger than the minimum required.

For purposes of illustration, consider an array of 45 modes, shown schematically in figure 2, at frequencies $m\delta(6 \leqslant m \leqslant 10)$ and spanwise wavenumbers $l = n\varepsilon(-4 \leqslant n \leqslant 4)$, for suitable constants δ and ε. We envisage that the central mode with $(m,n) = (8,0)$ has greatest amplitude, and that its resonant subharmonics, in the sense of Craik (1971), are at $(m,n) = (4,\pm 3)$: that is, we suppose that the latter modes have downstream wavenumber components exactly half that of the $(8,0)$ mode. This, of course, is still a crude representation of the Medeiros & Gaster data, with far fewer modes. The only available difference interactions with the required subharmonic frequency 4δ are those between members of the top and bottom rows; and only those with n-values differing by 3 give the correct spanwise wavenumber. Six such pairs are capable of driving a $(4,3)$ mode, and another six a $(4,-3)$ mode. With a broader range of available frequencies than the minimum range chosen here, the possible number of suitable interacting pairs is of course much increased. However, not all (or perhaps any) of these pairs need yield the desired subharmonic x-wavenumber.

In fact, the downstream wavenumber k is known to vary almost linearly with frequency; and it decreases only fairly slowly with l (or α_z) at fixed frequency, the change in k being approximately proportional to l^2 because of symmetry. For instance, at a Reynolds number R (based on displacement thickness) equal to 1000, the linearly unstable band of two-dimensional waves has dimensionless frequencies in the approximate range 0.051 to 0.122, with corresponding dimensionless wavenumbers varying from about 0.16 to 0.33. In contrast, with fixed frequency, the downstream wavenumber decreases only by about 10% as the angle of the wavenumber vector changes from 0 to 45°: see, for example, Craik (1971, figure 2) which shows a curve of constant frequency in the wavenumber plane at $R = 882$. An empirical formula in the spirit of our rough model is, in appropriate units,

$$k = 0.04 + 2.4\beta - 0.4l^2; \tag{4.1}$$

or, equivalently, $0.04 + 2.4m\delta - 0.4(n\varepsilon)^2$, where we may take $\delta = 0.0125$ and $\varepsilon = 0.06$ to correspond approximately with the packets of figure 1(*a*) at $x = 0.5\,\mathrm{m}$.

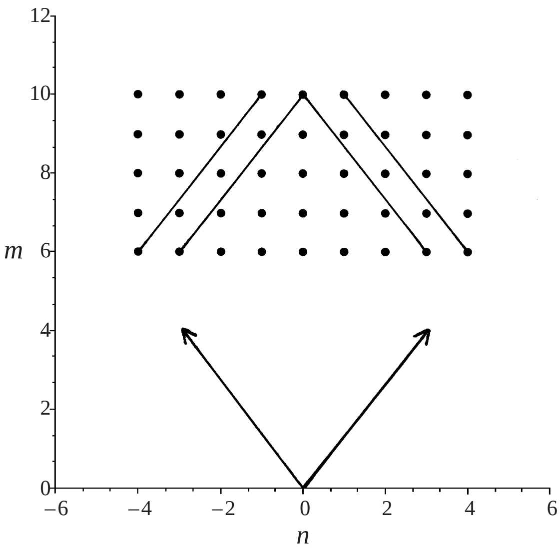

FIGURE 2. A postulated wavepacket with 45 components. The arrows denote subharmonics which resonate with the most-central mode, and lines indicate the interactions most likely to drive these harmonics.

With this model frequency–wavenumber relation (4.1), it is now easy to estimate the x-wavenumbers resulting from the various difference interactions at a fixed value of R. These must have the (m,n)-forms $(10,n)-(6,n-3)$ and $(10,n-3)-(6,n)(n=-1,0,1,2,3,4)$; and, according to our formula for k, their corresponding x-wavenumbers range from $k=0.098$ to 0.1416. Since the dominant $(8,0)$ mode has $k\equiv k_0=0.28$, only two pairs are really close to the resonant subharmonic modes $(4,\pm3)$ with $k=k_0/2=0.14$. These are the interactions of $(10,0)$ with $(6,3)$, and of $(10,0)$ with $(6,-3)$, which give $(4,\pm3)$ with $k=0.133$; and that of $(10,1)$ with $(6,4)$, and of $(10,-1)$ with $(6,-4)$, which give $(4,\pm3)$ with $k=0.1416$. The latter is particularly close to exact resonance.

This model readily extends to a *continuum* of modes occupying the same postulated rectangular region $0.075\leqslant\beta\leqslant0.125$, $-0.24\leqslant l\leqslant0.24$. Choose one component (mode A, say) with $\beta=\beta_A=0.125$, $l=l_A$; and another (mode B, say) with $\beta=\beta_B=0.075$, $l=l_B$. Their corresponding x-wavenumbers k_A and k_B are given by our empirical formula above. The resonant subharmonics of the central mode with $\beta=0.10$ and $l=0$ must have $k=0.14$, $l=0.18$ and $\beta=0.05$, as before. These resonant values coincide with the difference interaction of modes A and B whenever both $|l_B-l_A|=0.18$ and $k_A-k_B=0.14$. The latter is satisfied by our formula when $l_B^2-l_A^2=0.05$. The two solutions are approximately $l_A=\pm0.05$, $l_B=\pm0.23$ with corresponding signs. (In the above discrete model, $l=0.06n$ with $n=-1$, 0, 1, 2, 3, 4, pairs closest to resonance have $n_A=\pm1$, $n_B=\pm4$. The present values for precise resonance correspond to $n_A=\pm0.83$, $n_B=\pm3.83$ consistently with this result.) We may therefore conclude that a suitable excitation mechanism exists for 'seeding' the subharmonic mode, by direct quadratic interaction of members of the available set of wavemodes.

In the experiments of Medeiros & Gaster (1999*a,b*), the phase of the initial disturbance greatly affected the nonlinear growth. In these experiments, the localized initial disturbance was specified in such a way that the complex amplitudes a_j

of *all* excited wavemodes have the same phase, say ϕ. While these modes remain uninfluenced by nonlinearity, these phases remain constant as the corresponding A_j vary periodically with appropriate exponential factors $\exp(\mathrm{i}\omega_r t)$. A phase-dependent measure of the growth of the resonant subharmonic is given in (3.5), where the cosine term reduces to $|\cos[\frac{1}{2}\mathrm{ph}\lambda + \frac{1}{2}\phi - \mathrm{ph}\mu]|$. The greatest amplification therefore occurs when $\phi = 2\mathrm{ph}\mu - \mathrm{ph}\lambda$, and the least when $\phi = 2\mathrm{ph}\mu - \mathrm{ph}\lambda \pm \pi$. This least growth rate is zero according to our criterion with $p_1 \neq 0$; but, in theory, there is exponential decay from (2.4) when $p_1 = 0$. However, in practice, the theoretical prediction of decay when $\phi = 2\mathrm{ph}\mu - \mathrm{ph}\lambda \pm \pi$ is unlikely to be realised for long: rather, close to this phase relationship (which is called 'anti-resonant' by Bake *et al.* 2000), random external disturbances would come into play. But, for *almost* all initial phases, the evolution will be deterministic. Moreover, if $2\mathrm{ph}\mu - \mathrm{ph}\lambda$ happens to be small, positive initial disturbances will grow much faster than negative ones, as was found by Medeiros & Gaster (1999*a*, *b*).

It is known that, in some circumstances, $\mathrm{ph}\lambda$ is small. For instance, Hendricks (1975) reports computational results at $R = 882$. Then, the most unstable wave has downstream wavenumber $k_0 = 0.254$ and the resonant subharmonics have $k = 0.127$, $\pm l = 0.148$: the corresponding value of λ was found to be $6.0745 + 0.6499\,\mathrm{i}$. For other resonant triads with wavenumbers k in the range 0.2 to 0.5, Hendricks found that the phase of λ was always rather small, but it increased at values of k less than 0.2 (see Usher & Craik, table 2, p. 458). Unfortunately, corresponding quantitative results for the phase of μ are unavailable.

A more precise model than that given here must take account of the relative amplitudes and linear growth rates of all participating modes. A rather similar scenario should apply. In fact, many wave pairs of a continuum may excite disturbances close to the resonant subharmonics of the strongest two-dimensional mode, provided *the bandwidth of the packet of linearly unstable modes is sufficiently broad that:* (*a*) *its frequency range* $\omega_{max} - \omega_{min}$ *is at least as great as the subharmonic frequency, and* (*b*) *its range* $k_{max} - k_{min}$ *of downstream wavenumbers is as great as the downstream wavenumber of the resonant subharmonic.* If both these requirements are met, then the third requirement of a sufficient spread of spanwise wavenumbers is sure to be satisfied for initially localized disturbances. When exponential linear growth factors are retained, the forced quadratic terms involve exponentials equal to the sum of the growth rates of the two participating modes, as in result (2.2*a*): then, the initial growth of the driven terms would be even faster. But a more precise theoretical description would require extensive computation, for which the present writer has neither the appetite nor expertise. Nevertheless, the above simple considerations seem to provide a convincing explanation of the results of Medeiros & Gaster (1999*a*, *b*), which at first seemed so surprising.

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